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THE CHARACTERIZATION OF CLOCK BEHAVIOR WITH THE DYNAMIC ALLAN VARIANCE

L. Galleani[1], P. Tavella[2]
[1]Politecnico di Torino, C.so Duca degli Abruzzi 24, 10129 Torino, ITALY
[2]Istituto Elettrotecnico Nazionale (IEN), Strada delle Cacce 91, 10135 Torino, ITALY

Abstract – **We introduce the Dynamic Allan Variance, a quantity that characterizes the variation in time of the stability of an atomic clock. We connect the Dynamic Allan Variance to the Wigner spectrum, a time-frequency representation that can reveal the time-varying frequencies generally present in the clock error noise under nonstationary conditions. We also propose a practical implementation of the Dynamic Allan Variance for quasi-stationary clock noises, and we show numerical results that prove the validity of our approach, both on simulated and real data.**

I. INTRODUCTION

The noise characteristics of atomic clocks are generally estimated on a series of measurements that ideally are to be as long as possible, to allow a good estimation of the long term clock behavior. The common assumption for the clock behavior is to be stationary, i.e. with properties that do not vary with time, and consequently in the frequency domain. From experimental evidence we know that this is not always the case, mainly because of clock ageing in the long term and sudden anomalies in the short term. The aim of this paper is to present time and frequency domain statistical tools that are able to detect variations in the clock behavior, with the twofold purpose to characterize clock ageing and to evaluate the possibility to identify clock anomalies before they eventually show up.

To accomplish our goal we propose a new method, the Dynamic Allan Variance, an estimator that we show to be capable of representing the time variation of clock noises. We then derive the relation between the Dynamic Allan Variance and the Wigner spectrum, a quantity adopted in time-frequency analysis [1] to represent the time variation of the frequencies of a stochastic process.

For the case of quasi-stationary clock behaviors we propose a practical implementation of the Dynamic Allan Variance, that we then apply to real and simulated atomic clock data, showing the effectiveness of the approach.

II. A REVIEW OF THE ALLAN VARIANCE

The Allan variance [2] has been introduced to characterize the frequency stability of oscillators. If we write the signal (e.g. the voltage) generated by an oscillator as (we follow [3] for the notation)

$$u(t) = (U_0 + \varepsilon(t))\sin(2\pi\upsilon_0 t + \phi(t)) \tag{1}$$

we see how the nominal values of amplitude and frequency should be U_0 and υ_0, but due to the random fluctuations $\varepsilon(t)$ and $\phi(t)$ the signal $u(t)$ differs from the ideal case. While the amplitude fluctuation $\varepsilon(t)$ can be neglected, the phase factor $\phi(t)$ plays an important role in the characterization of precise oscillators. The instantaneous frequency of the signal $u(t)$ is

$$\upsilon(t) = \upsilon_0 + \frac{1}{2\pi}\frac{d\phi}{dt} \tag{2}$$

and this allows us to define the so called fractional frequency offset as

$$y(t) = \frac{\upsilon(t) - \upsilon_0}{\upsilon_0} \tag{3}$$

The quantity $y(t)$ allows the study of the frequency stability of an oscillator independently from the actual value of the nominal frequency υ_0. The corresponding normalized phase fluctuation $x(t)$ is introduced as

$$y(t) = \frac{dx}{dt} \tag{4}$$

Both $x(t)$ and $y(t)$ can be modeled as stochastic processes, and their variation in time is a measure of the stability of the oscillator.

The standard quantity that defines the stability of an oscillator is the Allan variance [2], that is defined as the time average of the N-sample variance

$$\left\langle \sigma_y^{\,2}(N,T,\tau) \right\rangle = \left\langle \frac{1}{N-1}\sum_{k=1}^{N}\left(\bar{y}_k - \sum_{l=1}^{N}\bar{y}_l\right)^2 \right\rangle \tag{5}$$

where N is the number of considered samples, τ is the measurement time and T is the interval between every measurement. The k-th measurement of the mean frequency is

$$\bar{y}_k(t) = \int_{t_k}^{t_k+\tau} y(t)dt = \frac{x(t_k+\tau) - x(t_k)}{\tau} \tag{6}$$

Hence the Allan variance is the time average of a quantity defined for discrete times t_k, and therefore it can be written as

$$\left\langle \sigma_y^{\,2}(N,T,\tau) \right\rangle = \lim\nolimits_{M\to\infty} \sum_{i=1}^{M} \sigma_y^{\,2}(N,T,\tau;i)$$

(7)

where the i index indicates the generic N-sample variance.

When N=2 and T=τ we have the well known 2-sample Allan variance without dead-time

$$\sigma_y^2(\tau) = \left\langle \sigma_y^2(2,\tau,\tau) \right\rangle = \tfrac{1}{2}\left\langle \left(\overline{\Delta}(t_{k+1},\tau)\right)^2 \right\rangle \tag{8}$$

where

$$\overline{\Delta}(t_k,\tau) = \overline{y}_{t_k+\tau} - \overline{y}_{t_k} \tag{9}$$

is the so called "second phase difference". The Allan variance is widely used for it converges (as the N-sample variance) for all the typical random processes that one encounters in the frequency fluctuation *y(t)* of real clocks (also it is very simple to compute).

The Allan variance has a very easy connection to the frequency spectrum of the random process *y(t)*

$$\sigma_y^2(\tau) = 2\int_0^{+\infty} S_y(f) \left|H_A(f;\tau)\right|^2 df \tag{10}$$

where

$$\left|H_A(f;\tau)\right|^2 = \left(\frac{\sin^2 \pi f\tau}{\pi f\tau}\right)^2 \tag{11}$$

Here we present only the connection to the 2-sample case, but the same relation holds for the N-sample variance, just substituting the appropriate transfer function $H_A(f;\tau)$.

It is clear from the definition (8) of the Allan variance that we require the stationarity of the stochastic process $\overline{\Delta}(t_k,\tau)$, or equivalently of the increments between couples of consecutive frequency values $\overline{y}(t_k), \overline{y}(t_{k+1})$. The stationarity guarantees the convergence of the Allan variance.

III. THE DYNAMIC ALLAN VARIANCE

We are now interested in defining a measure of stability of an oscillator that does not make any assumption on the stationarity of the increment process $\overline{\Delta}(t_k,\tau)$. To do so we propose the definition of the new quantity

$$\sigma_y^2(t,\tau) = \frac{1}{2} E\left[\overline{\Delta}(t,\tau)^2\right] \tag{12}$$

where E[ ] stands for the expectation value. The quantity $\sigma_y^2(t,\tau)$, due to the ensemble averaging of (12), becomes now a function of both time t and the observation interval τ. We point out the following considerations on this new definition of the Allan variance, that we call the DAVar (Dynamic Allan Variance)

- It is an ensemble average of the increment process $\overline{\Delta}(t,\tau)$;
- Time is continuous in the increment process $\overline{\Delta}(t,\tau)$ and hence also in the DAVar, $\sigma_y^2(t,\tau)$;
- There are no assumptions on the random process *y(t)*, nor on the increment process $\overline{\Delta}(t,\tau)$; the only assumption is obviously the convergence of (12).

Hence the DAVar can be used to study the stability of oscillators whose properties change with time, thus producing nonstationary phase and frequency fluctuations that cannot be properly estimated with classical tools of analysis.

Since one commonly represents deviations, we also introduce the Dynamic Allan Deviation (DADev)

$$\sigma_y(t,\tau) = \frac{\sqrt{2}}{2}\sqrt{E\left[\overline{\Delta}(t,\tau)^2\right]} \tag{12}$$

A. Stationary Case

When the increment process $\overline{\Delta}(t,\tau)$ is stationary, then the ensemble average (12) can be computed as a time average. The DAVar is not a function of time anymore, and it collapses into the standard Allan variance

$$\sigma_y^2(t,\tau) = \frac{1}{2}E\left[\overline{\Delta}(t,\tau)^2\right] = \frac{1}{2}E\left[\overline{\Delta}(\tau)^2\right] = \frac{1}{2}\left\langle \overline{\Delta}(\tau)^2 \right\rangle \equiv \sigma_y^2(\tau) \tag{13}$$

B. Quasi-Stationary Case

When the increment process $\overline{\Delta}(t,\tau)$ varies very slowly with time, we can approximate the DAVar by computing its value on every quasi-stationary interval, just by using the usual Allan variance estimator.

For such situations we have implemented an estimator of DAVar that is in fact just a "sliding" Allan variance estimator. The algorithm allows to select the length of every "slice" of the process *y(t)*, and also the values of time t where it has to be computed. The result is an approximation to the real DAVar. In Sect. V we present some examples obtained using the described algorithm.

IV. CONNECTION TO TIME-FREQUENCY ANALYSIS

An important connection exists between the Allan Variance and the classical Fourier spectrum of the random process *y(t)* (or the phase offset *x(t)*), as reviewed in Sect. II. One may therefore wonder whether it is possible to connect the DAVar to a representation of *y(t)* that can reveal its possible time-varying frequency structure, a consequence of the nonstationarity assumption made.

We have been able to find such connection, and in order to show the obtained result we first need to introduce a proper representation of the instantaneous frequency spectrum of a nonstationary process. In general, in fact, we expect the frequency spectrum to be a function of time for an arbitrary random process $y(t)$. Here we adopt the definition proposed in time-frequency analysis, a field of signal analysis that investigates the concept of time-varying spectra [1]. In time-frequency Analysis the concept of time-frequency distribution $P(t,\omega)$ (for the frequency here we adopt $\omega = 2\pi f$ as is often done in signal analysis) is introduced, and several (infinite in general) distributions can be introduced to define the idea of a time-varying spectrum. In particular here we adopt the Wigner spectrum as the definition for the time-varying spectrum of $y(t)$.

The Wigner spectrum is defined as [4]

$$\overline{W}_y(t,\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}E\left[y^*(t-\tau/2)y(t+\tau/2)\right]e^{-i\tau\omega}d\tau \qquad (14)$$

and it is basically the expectation value of the well known Wigner distribution [5]

$$W_y(t,\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}y^*(t-\tau/2)y(t+\tau/2)\,e^{-i\tau\omega}d\tau \qquad (15)$$

(the star indicates complex conjugation), defined for a deterministic signal $y(t)$.

The Wigner spectrum satisfies a mathematical property that allows a simple link to the DAVar, that is, by taking its frequency marginal we get [1]

$$\int_{-\infty}^{+\infty}\overline{W}_y(t,\omega)d\omega = E\left[|y(t)|^2\right] \qquad (16)$$

This allows to immediately obtain a link to the Dynamic Allan Variance, since we have

$$\sigma_y^{\,2}(t,\tau)=\frac{1}{2}E\left[\overline{\Delta}(t,\tau)^2\right]=\frac{1}{2}\int_{-\infty}^{+\infty}\overline{W}_{\overline{\Delta}(t,\tau)}(t,\omega)d\omega \qquad (17)$$

Now, since it holds

$$\overline{\Delta}(t,\tau)=h_A(t;\tau)*y(t)=\int_{-\infty}^{+\infty}h_A(t-t';\tau)y(t')dt' \qquad (18)$$

where the star here indicates convolution and $h_A(t;\tau)$ is the inverse Fourier transform of (11). Now the Wigner spectrum of the convolution of a deterministic signal and a stochastic process satisfies

$$\begin{aligned}\overline{W}_{\overline{\Delta}}(t,\omega;\tau)&=\overline{W}_{h_A*y}(t,\omega;\tau)\\ &=2\pi\int_{-\infty}^{+\infty}W_{h_A}(t-t',\omega;\tau)\overline{W}_y(t',\omega)dt'\end{aligned} \qquad (19)$$

where $W_{h_A}(t,\omega;\tau)$ is the Wigner distribution of the window $h_A(t;\tau)$. Substituting (19) into (17) we finally obtain

$$\sigma_y^{\,2}(t,\tau)=\pi\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}W_{h_A}(t-t',\omega;\tau)\overline{W}_y(t',\omega)dt'd\omega \qquad (20)$$

which is the desired link between the DAVar and the time-frequency spectrum of the random fluctuation $y(t)$.

We point out that the Wigner distribution of the window $h_A(t;\tau)$ can be easily evaluated, but we will do so in a forthcoming publication where we will analytically study several scenarios of nonstationary random fluctuations $y(t)$ with the DAVar. In the next section we instead show some numerical results.

V. EXAMPLES

We now present two numerical simulations, both considering a nonstationary phase noise. We evaluate the DAVar for both the two cases, and show how it can represent in a very efficient and easy to understand way the nonstationarity of the studied signals.

A. Sudden Variation

We consider a phase offset $x(t)$ that is made by piecewise white noise with different variance on each disjoint time interval. In particular we make the following choice

$$\sigma_y^{\,2}=\begin{cases}1 & 0\le t\le 475\,s\\ 5 & 475\le t\le 525\,s\\ 1 & 525\le t\le 1000\,s\end{cases} \qquad (21)$$

A typical pattern of $x(t)$ with such type of noise is reported in Fig. 1

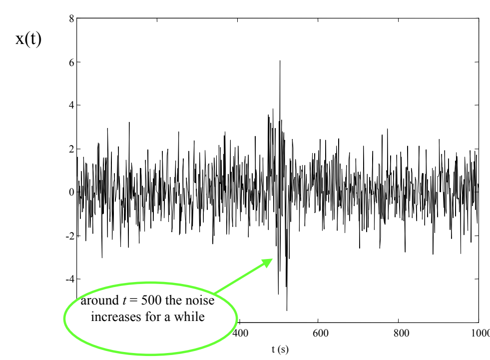

Figure 1. A series of $x(t)$ affected by white noise with variance given by (21).

In Fig. 2 we show the DADev computed using the algorithm described in Sect. III.B. The picture clearly reveals the sudden variation that the white PM noise is undergoing in the middle of the time axis. The length of the time window used in the algorithm is 101 s. In Fig. 3 we show the classical Allan deviation of the same data, and we notice how the sudden variation is hidden in that representation.

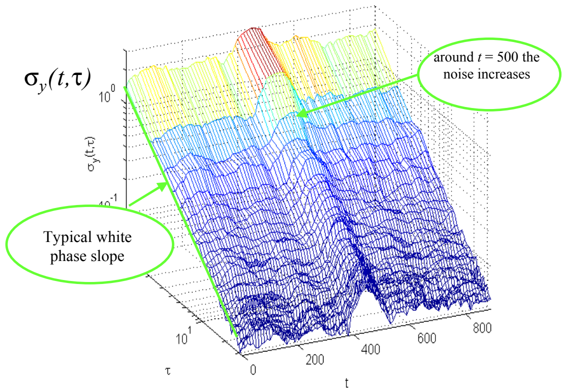

Figure 2. Dynamic Allan Deviation of *x(t)* built with a white PM noise that suddenly undergoes a "jump" in the variance (the model is (21)). The DAVar estimator clearly tracks the nonstationarity.

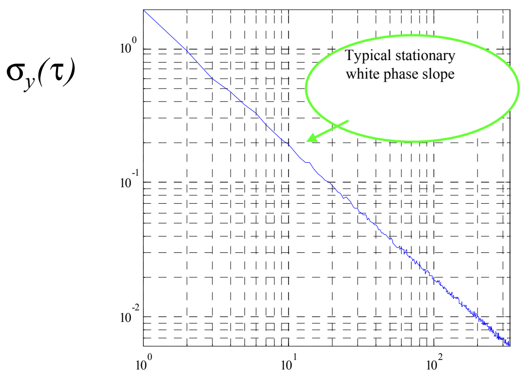

Figure 3. Allan deviation of the entire series of *x(t)* from (21).

B. Two-Level White PM

Now we consider a random fluctuation in *x(t)* that is made by two white PM noises with different variances. The model for *x(t)* is a white Gaussian noise with variance given by

$$\sigma_y^{\ 2} = \begin{cases} 1 & 0 \le t < 500\,s \\ 10 & 500 \le t \le 1000\,s \end{cases} \tag{22}$$

A typical pattern of *x(t)* with such type of noise is reported in Fig. 4

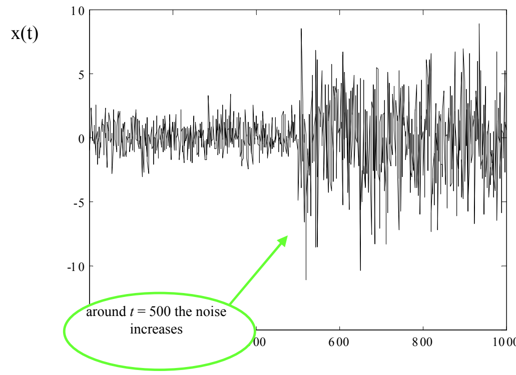

Figure 4. A series of the white noise *x(t)* with variance given by (22).

In Fig. 5 we show the corresponding Dynamic Allan Deviation computed using the usual algorithm described in Sect III.B. It is again easy to see how the estimator tracks the variation of the variance that happens in *x(t)* at half of the signal duration.

We then show an interesting comparison in Fig. 6 and Fig. 7, respectively between the Allan deviation of *x(t)* built from (22) and the Allan deviation of another time series $x_2(t)$ made by a white PM noise with a constant variance equal to 5, practically the average of the two values used to build the previous *x(t)* series. It is interesting to see how the two pictures are very close to each other. This suggests the intuitive idea that different phase deviations *x(t)* have the same Allan variance, but different Dynamic Allan Variance.

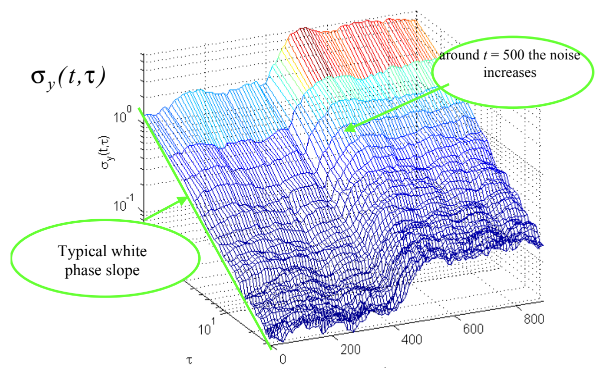

Figure 5. Dynamic Allan Deviation of *x(t)* from (22). The signal is built using two subsequent white PM noises with different variances. The DAVar estimator well represents the nonstationarity of *x(t)*.

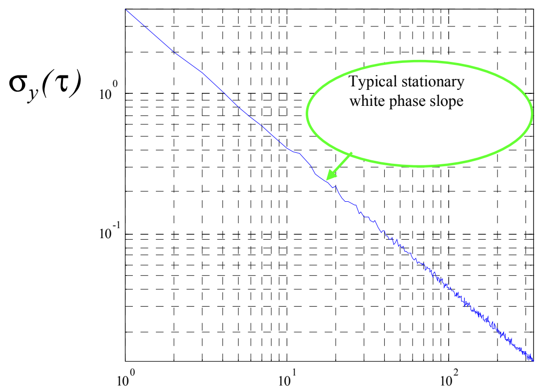

Figure 6. Allan deviation of *x(t)* from (22). The nonstationarity in *x(t)* is hidden in this representation.

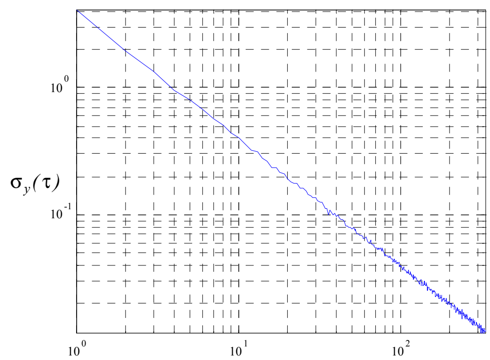

Figure 7. Allan deviation of a process $x_2(t)$ that is a white PM noise with a variance that is the average of the two variances used to build *x(t)* shown in Fig. 6. Notice that the two deviations (Fig. 6 and Fig.7) are very similar.

C. Two different noises in sequence

We now consider a random fluctuation on *x(t)* that changes during the clock life from a white PM to a white FM (giving a random walk on the phase *x(t)*). A realization is reported in Fig. 8

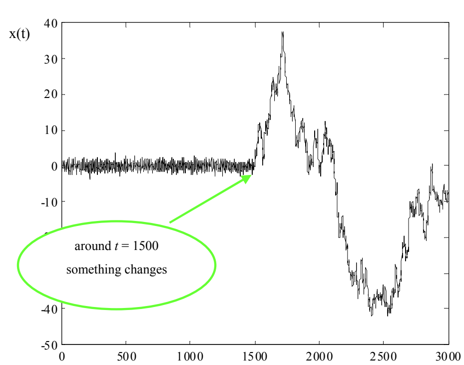

Figure 8. Typical pattern of a clock phase deviation *x(t)* that is affected by a white PM noise that suddenly changes into a white FM, giving raise to a random walk on the phase.

In Fig. 9 we show the corresponding Dynamic Allan Deviation. It is again easy to see how the estimator tracks the change in variance that happens in *x(t)* at half of the signal duration.

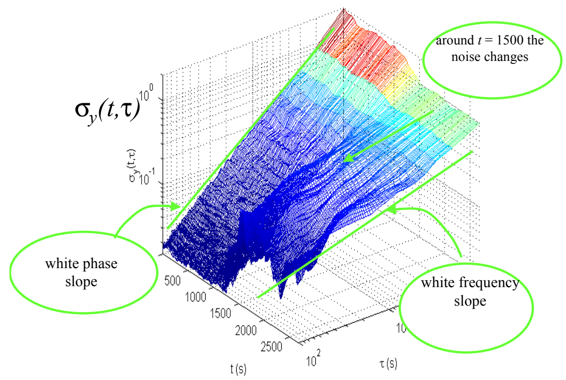

Figure 9. Dynamic Allan Deviation of *x(t)* from Fig.8. The signal is built using two subsequent types of noise: white PM and white FM. The DAVar estimator well represents the nonstationarity of *x(t)*.

In Fig. 10 the Allan deviation of the entire series of *x(t)* reported in Fig. 8 is depicted together with the slopes corresponding to pure white PM and pure white FM. We see that the experimental results tend to indicate that the dominant noise is a white FM, completely occulting the presence of the subsequent types of noise.

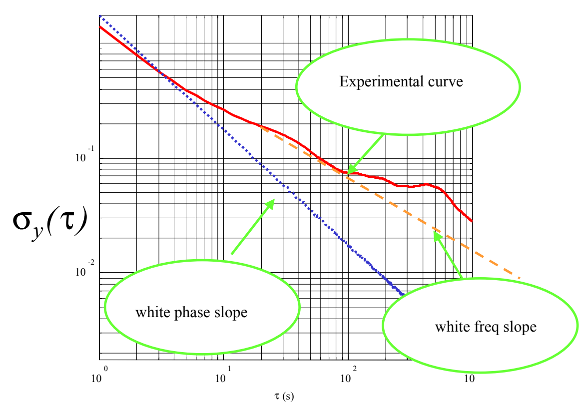

Figure 10: Allan deviation of the entire series of *x(t)* from Fig. 8., together with the slopes corresponding to pure white PM and pure white FM. The nonstationarity of *x(t)* is hidden in this representation.

As a last example, we report in Fig. 11 the Dynamic Allan Deviation obtained from a series of real measures obtained from the IEN Cesium fountain primary frequency standard. From this preliminary analysis the presence of a periodic spurious signal appears.

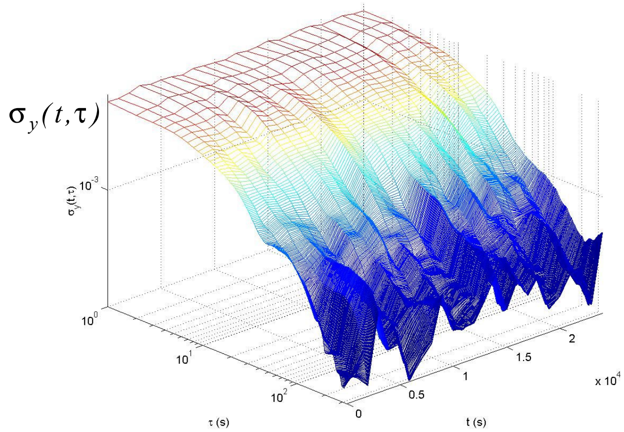

Figure 9. Dynamic Allan Deviation obtained from a series of real measures from the IEN Cesium fountain primary frequency standard. The presence of a periodic spurious signal is highlighted by the method.

VI. CONCLUSION

We have presented a new quantity, the Dynamic Allan Variance, whose objective is to represent the variation of clock noises when no stationarity assumption is made. This means that the Dynamic Allan Variance can be used for an arbitrary random fluctuation *x(t)*. We have also linked the DAVar to the Wigner spectrum of *x(t),* a quantity that represents the instantaneous spectrum of a random process with an arbitrary nonstationary nature. The link is a very neat expression that we believe will allow us to analytically characterize nonstationary fluctuations in *x(t)*.

We also propose an algorithm to estimate the DAVar for quasi-stationary signals *x(t),* and by showing numerical examples on both simulated and real data, we prove how the DAVar can clearly reveal the time variation of the statistical properties of the clock noise fluctuations. A connection between the DAVar and the wavelet representation is also under investigation, as well as the application of the time-frequency spectral analysis to atomic clock data [6].

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